On the Statistics of Many Interacting Waves far from Thermal Equilibrium

W. Wonneberger and J. Lempert

Fakultät für Physik der Universität Freiburg i. Br., Germany

The intensity distribution function $P(I)$ for the incoherent superposition of interacting stationary waves far from thermal equilibrium and coupled to many other waves e. g. in active photon and phonon systems is determined analytically for a specific interaction model. The model consists of $N_m \gg 1$ equivalent waves concentrated in a small frequency band which interact through their intensities alone. $P(I)$ refers to $N_m'' < N_m$ of these waves. The remaining $N_m' = N_m - N_m''$ waves then act as a reservoir. $P(I)$ is shown to be asymptotically given by a Beta-distribution for $N_m' \gg 1$. It is found that the interactions thermalize each individual wave which otherwise would be laser-like concerning its statistical properties. The photon counting distribution associated with $P(I)$ is also discussed. For N_m'' comparable to N_m, it can differ significantly from the photon counting distribution of N_m'' independent thermal waves.

1. Introduction

A nonlinear amplification mechanism can stabilize a single wave or a whole group of waves at a level far from thermal equilibrium. This leads to self sustained oscillations which fluctuate around a stationary state. They can be realized e. g. in active photon (cf. [1]) and active phonon (cf. [2]) systems. The statistical properties of a single wave have been thouroughly discussed in the last years especially in conjunction with the laser. It is impossible to cite all the important contributions to this field. The reader is referred to the reprint collection [3], to the two comprehensive works [1, 4] and to the article in [5].

In the case of several waves, the problem becomes complicated due to the interactions between the waves. Far from thermal equilibrium, interactions will always be present between several waves and cannot be neglected. Using dissipation-fluctuation theorems some statistical properties of the two mode laser have been discussed in [6, 7]. Increasing numbers N_m of waves still complicate the problem until for $N_m \gg 1$ asymptotic expansions may prove useful to analyse at least some simple questions concerning the statistics of many interacting waves.

We have investigated a simple interaction model of a great number $N_m \gg 1$ of equivalent waves concentrated in a small frequency band. Each wave interacts with every other wave in the same way and through their respective intensities alone. This "normal multimode" type of interaction (cf. [4]) appears e. g. in a RPA theory of nonlinear acoustoelectric

Reprint requests to Dr. W. Wonneberger, Fakultät für Physik der Universität Freiburg i. Br., *D-7800 Freiburg i. Br.*, Hermann-Herder-Str. 3.

noise amplification [8]. The associated $(2 N_m)$-dimensional Fokker Planck equation (Section 2) then is a straight forward extension of the corresponding single wave Fokker Planck equation [9] and its stationary intensity distribution function P is easily obtained by direct integration. It is a function of the N_m intensity variables of the waves. The distribution function P is a (classical) P-representation (cf. e. g. [10]) of a density operator ϱ appropriate for a specific situation far from thermal equilibrium. As it stands, P has no direct physical relevance, except for the possibility to compute certain averages or restricted probability distributions P_c from P.

We have determined the intensity distribution function $P(I)$ and the generating function $Q(\lambda)$ for the incoherent superposition I of $N_m'' < N_m$ waves (Sections 3 and 4). From our result for $Q(\lambda)$ we have derived and discussed a recurrence relation for the photon counting distribution $p(n)$ appropriate for short counting intervals (Section 5). A definite example where these results are of interest is a photon counting experiment of Brillouin scattered light from amplified vibrational flux [11]. The Brillouin scattered light of a highly stabilized laser then permits to study the statistics of N_m'' vibrational waves which are "dressed" by the remaining $N_m' = N_m - N_m''$ waves contained in the frequency band of the N_m interacting waves. The momentum selection rule in the photon-phonon interaction and the focusing of the laser beam permit to vary the number N_m''. In this specific example one has $N_m \gtrsim 10^5$ and the noise temperature of each wave is about $10^{10} T_0$.

As will become clear from the mathematical formulations in Section 2 our problem is more general

than indicated by this example. This is analogous to the single wave case which according to [12] appropriately accounts for a wide class of amplitude stabilized waves (self sustained oscillations) subjected to small fluctuations. Furthermore, our results seem to contain a general property of many interacting waves far from thermal equilibrium namely that the interactions thermalize each individual wave which without interactions would be amplitude stabilized.

We therefore present the following results in a more abstract way. A short account of them has already been given in [13].

2. Formulation of the Problem

Our starting point is a one-dimensional stochastic wave field $u(x, t)$. We suppose, that $u(x, t)$ permits the spectral decomposition

$$u(x, t) = \sum_{Q > 0} \tilde{u}(Q, t) \exp\{-i \omega(Q) t + i Q x\} + \text{c. c.} \tag{1}$$

appropriate for a spatially homogeneous situation. The Fourier components are supposed to undergo stochastic processes defined by Langevin equations of the form

$$\dot{\tilde{u}}(Q) = \alpha(Q) \tilde{u}(Q) + \tilde{\Gamma}(Q) \tag{2}$$

where $\tilde{\Gamma}(Q)$ are uncorrelated complex Markoffian fluctuating forces with the strength *

$$\langle \tilde{\Gamma}(Q, t) \tilde{\Gamma}^*(Q', t') \rangle = 4 q(Q) \delta_{Q, Q'} \delta(t - t') \tag{3}$$

and with no phase preferences. $\alpha(Q)$ are gain functions which in general depend on the whole set $\{\tilde{u}(Q')\}$ rendering the waves $\tilde{u}(Q)$ statistically dependent. This should be contrasted to the well-known statistical description of a single wave

$$u(t) = \sum \tilde{u}(Q) \exp\{-i \omega(Q) t\}$$

where $\tilde{u}(Q)$ are independent stochastic variables with specified distribution functions.

q is assumed to be independent of $\{\tilde{u}(Q')\}$. The determination of q usually is a quantum mechanical task. Equations like (2) appear e. g. in the statistical theory of the laser (cf. [4, 5]) or of nonlinear acousto-electric noise amplification [14]. They usually result when the wave equations are treated in the rotating

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* Where it seems useful we follow the notation of [5].

wave approximation (RWA) and adiabatic elemination procedures are employed (cf. [4, 5]).

The general method of treating such a problem is now well-known [15]. Instead of the set of nonlinear differential Eqs. (2) the associated Fokker-Planck equation for the multidimensional transition probability P is considered. Using the correspondences

$$Q \triangleq \nu, \ \tilde{u}(\nu) \triangleq \{\text{Re } \tilde{u}(\nu); (1/i) \text{ Im } \tilde{u}(\nu)\} = \boldsymbol{u}_\nu$$

the Fokker Planck equation reads

$$\dot{P} = -\sum_\nu \nabla_\nu [\alpha(\nu) \boldsymbol{u}_\nu P] + \sum_\nu q(\nu) \Delta_\nu P \tag{4}$$

with $\qquad \nabla_\nu = \text{div } \boldsymbol{u}_\nu, \ \Delta_\nu = \text{div } \boldsymbol{u}_\nu \text{ grad } \boldsymbol{u}_\nu.$

We are interested in the stationary solution of Eq. (4) and its consequences. Though the formal determination of the stationary solution may be discussed on a very general basis [16, 17] we prefer to specialize to a case which can be solved analytically in all details. Our case is characterized by:

1. The gains α depend only on the intensities $\{\tilde{u}\tilde{u}^*\}$. This is usually a consequence of a random phase (RPA) assumption in the determination of α.

2. The fluctuations around the stationary state or "operating point" are small enough to permit the truncated Taylor series expansion

$$\alpha(\nu) = \sum_{\nu'} \Delta\alpha(\{(\tilde{u}\tilde{u}^*)_{\text{op}}\}; \nu, \nu') \tag{5}$$
$$\cdot [1 - \tilde{u}(\nu') \tilde{u}^*(\nu') / (\tilde{u}(\nu') \tilde{u}^*(\nu'))_{\text{op}}].$$

The operating point is given by a set of intensity values $\{(\tilde{u}(\nu) \tilde{u}^*(\nu))_{\text{op}}\}$ where $(\tilde{u}(\nu) \tilde{u}^*(\nu))_{\text{op}}$ must be determined from self consistency requirements which in our case turn out to be rather simple (Section 4).

3. All N_{m} waves are concentrated in a small wave number band around $\overline{Q}$ such that no essential Q dependencies other than those on $\overline{Q}$ appear.

α then takes on the simple form

$$\alpha = \Delta\alpha \sum_{\nu'=1}^{N_{\text{m}}} (1 - \tilde{u}(\nu') \tilde{u}^*(\nu') / (\tilde{u}\tilde{u}^*)_{\text{op}}) \tag{6}$$

and the multimode Fokker Planck equation permits the same scaling procedure as in the single mode case [5, 9]. With the transformations

$$\nabla_\nu \to \sqrt[4]{\Delta a / [(\tilde{u}\tilde{u}^*)_{\text{op}} q]} \nabla_\nu,$$
$$\partial/\partial t \to \sqrt{\Delta\alpha q / (\tilde{u}\tilde{u}^*)_{\text{op}}} \cdot \partial/\partial t, \tag{7}$$
$$\boldsymbol{u}_\nu \to \sqrt[4]{(\tilde{u}\tilde{u}^*)_{\text{op}} q / \Delta\alpha} \boldsymbol{u}_\nu$$

and the abbreviation

$$a = \sqrt{(\tilde{u}\tilde{u}^*)_{\text{op}} \Delta\alpha / q} \tag{8}$$

we get

$$\dot{P} = - \sum_{\nu=1}^{N_m} \nabla_\nu \left(\sum_{\nu'=1}^{N_m} (a - |\boldsymbol{u}_{\nu'}|^2) \, \boldsymbol{u}_\nu \, P \right) + \sum_{\nu=1}^{N_m} \Delta_\nu P .$$

(9)

For the following only the relations between the true intensities $\tilde{u}(\nu) \, \tilde{u}^*(\nu)$ and the scaled new intensities z_ν are of importance:

$$\tilde{u}(\nu) \, \tilde{u}^*(\nu) = \sqrt{(\tilde{u} \, \tilde{u}^*)_{op} \, q / \Delta \alpha} \; z_\nu = [(\tilde{u} \, \tilde{u}^*)_{op}/a] \, z_\nu .$$

(10)

Introducing

$$V = \tfrac{1}{4} \left(\sum_{\nu=1}^{N_m} (z_\nu - a) \right)^2$$

(11)

we have in the new variables

$$\dot{P} = \sum_\nu \operatorname{div}_\nu \big((\operatorname{grad}_\nu V) \, P \big) + \sum_\nu \operatorname{div}_\nu \operatorname{grad}_\nu P$$

(12)

giving the stationary intensity distribution

$$P(\{z_\nu\}) \propto \exp\{-V\} = \exp\left\{ -\tfrac{1}{4} \left(\sum_{\nu=1}^{N_m} (z_\nu - a) \right)^2 \right\}.$$

(13)

The normalization is not of interest at this stage of the calculations. As in the case of a single mode laser near threshold [5] the quantity a is of the order of unity for highly amplified stationary vibrational flux of about $10^{10} \, T_0$ noise temperature [18].

Equation (13) is the basis of our investigation. It represents the unnormalized distribution function for the scaled intensities of N_m equivalent waves far from thermal equilibrium which interact via their intensities rendering the waves statistically dependent.

According to Section 1 we are interested in the distribution function $P(I)$ of the quantity

$$I = \sum_{\mu=1}^{N_m''} z_\mu$$

(14)

which is the intensity due to the incoherent superposition of $N_m'' < N_m$ individual waves. We take the incoherent superposition as an a priori property of the detected signal composed of N_m'' waves.

3. Determination of $P(I)$ and $Q(\lambda)$

The determination of $P(I)$ proceeds in a two step way. We calculate from P the contracted distribution function $P_c(\{z_\mu\}, \mu = 1, \ldots, N_m'')$ by integrating P over $N_m' = N_m - N_m''$ of its variables using the method of steepest descent and then compute the generating function $Q(\lambda) = \langle \exp\{-\lambda I\} \rangle$ from P_c under an allowed approximation. $Q(\lambda)$ turns out to be associated with the Beta-distribution in Statistical Mathematics (cf. [19]).

a) Determination of P_c

We do not need the normalization neither of P nor of P_c. Only in the very last step of our calculations, the normalization of $P(I)$ will be determined. We refer to the class of unnormalized distribution functions by employing a $\triangleq$ sign in the equations. This class remains unchanged by adding or dropping any factor independent of z_ν. Consequently one has

$$P_c(\{z_\mu\}) \triangleq \int_0^\infty \prod_{\nu=N_m''+1}^{N_m} dz_\nu \times \exp\left\{ -\lambda_1^2 - \lambda_1 \sum_\nu (z_\nu - a) - \tfrac{1}{4} \left(\sum_\nu (z_\nu - a) \right)^2 \right\}.$$

(15)

Here, we have defined

$$\lambda_1 = \tfrac{1}{2} \sum_{\mu=1}^{N_m''} (z_\mu - a).$$

(16)

Substituting $z_\nu = z_\nu' + a$ setting $\mu_1 = \Sigma z_\nu'$ and using the relation

$$\delta \left(\mu_1 - \sum_{\nu=N_m''+1}^{N_m} z_\nu' \right) = (1/2\pi) \int_{-\infty}^\infty dt \exp\{-i(\mu_1 - \Sigma z_\nu') \, t\}$$

(17)

we find [**]

$$P_c(\{z_\mu\}) \triangleq (1/2\pi) \int_{-N_m'a}^\infty d\mu_1 \int_{-\infty}^\infty dt \exp\{-\lambda_1^2 - \lambda_1 \mu_1 - \tfrac{1}{4} \mu_1^2 - i \mu_1 t\} \prod_\nu \int_{-a}^\infty dz_\nu' \exp\{i z_\nu' t\} .$$

(18)

Shifting the t-integration contour slightly into the upper half plane: $t \to t + i\varepsilon, \varepsilon > 0$ we can easily evaluate the z_ν' integrations. Each of them gives

$$\int_{-a}^\infty dz' \exp\{i z' t\} = (i/(t + i\varepsilon)) \exp\{-i a t\} .$$

(19)

[**] The authors are indebted to Dr. K. Dettmann for pointing out to them this integration method.

Closing the t-integration contour at infinity in the lower half plane gives no contribution because of $\mu_1 + N_{\mathrm{m}}' a = \Sigma z_\nu > 0$. The result of the t-integration therefore is given by the contribution of the single pole of order N_{m}' at $t = -i\varepsilon$:

$$
P_{\mathrm{c}}(\{z_\mu\}) \triangleq \int\limits_{-N_{\mathrm{m}}' a}^{\infty} \mathrm{d}\mu_1 \exp\{-\lambda_1{}^2 - \lambda_1 \mu_1 - \tfrac{1}{4}\mu_1{}^2\} \oint \mathrm{d}t \; \frac{\exp\{-i(\mu_1 + N_{\mathrm{m}}' a)\, t\}}{(t + i\varepsilon)^{N_{\mathrm{m}}'}}
$$

$$
\triangleq \int\limits_{-N_{\mathrm{m}}' a}^{\infty} \mathrm{d}\mu_1 \exp\{-\lambda_1{}^2 - \lambda_1 \mu_1 - \tfrac{1}{4}\mu_1{}^2 + (N_{\mathrm{m}}' - 1)\ln(\mu_1 + N_{\mathrm{m}}' a)\}. \tag{20}
$$

In this way, the problem of contracting the distribution function P over N_{m}' of its variables is reduced to a single integration in Equation (20). Here, for the first time, we take advantage of the many wave aspect of our problem. $N_{\mathrm{m}}' - 1$ is assumed to be a number much greater than unity and the method of steepest descent can be employed in evaluating the integral (20).

Setting

$$
f(\mu_1) = -\lambda_1 \mu_1 - \tfrac{1}{4}\mu_1{}^2 + (N_{\mathrm{m}}' - 1)\ln(\mu_1 + N_{\mathrm{m}}' a) \tag{21}
$$

the relevant saddle point is determined from $f'(\mu_1) = 0$ to lie at

$$
\mu_{10} = -(\lambda_1 + N_{\mathrm{m}}' a/2) + \sqrt{(\lambda_1 - N_{\mathrm{m}}' a/2)^2 + 2(N_{\mathrm{m}}' - 1)}. \tag{22}
$$

Conveniently the integration along the real axis is the optimal pass through the maximum at this point and one has for $N_{\mathrm{m}}' \gg 1$, $N_{\mathrm{m}}'' \ll N_{\mathrm{m}}'^2$

$$
P_{\mathrm{c}}(\{z_\mu\}) \triangleq \left(\varepsilon + \sqrt{\varepsilon^2 + 2(N_{\mathrm{m}}' - 1)}\right)^{N_{\mathrm{m}}' - 1} \exp\{-\tfrac{1}{2}\varepsilon^2 + \tfrac{1}{2}\varepsilon \sqrt{\varepsilon^2 + 2(N_{\mathrm{m}}' - 1)}\} \tag{23}
$$

where $\varepsilon = N_{\mathrm{m}}' a/2 - \lambda_1 = (N_{\mathrm{m}} a - \Sigma z_\mu)/2$. This is the desired result for the contracted distribution P_{c}. A discussion of the validity of the method of steepest descent in our case is contained in the Appendix I.

b) Determination of $P(I)$ and $Q(\lambda)$

In principle, it is possible to derive the normalized distribution function $P(I)$ from Eq. (23) by averaging the quantity $\delta(I - \Sigma z_\mu)$. I is defined by Equation (14). We prefer, however, to compute the generating function

$$
Q(\lambda) = \langle \exp\{-\lambda I\} \rangle \tag{24}
$$

at first and then deduce the form of $P(I)$ leading to $Q(\lambda)$.

In using the contracted distribution P_{c} we encounter the same normalization problem described at the beginning of Section 3. Any P_{c} according to Eq. (23) will yield an unnormalized generating function $Q_{\mathrm{u}}(\lambda)$. The correct generating function $Q(\lambda)$ is then obtained by

$$
Q(\lambda) = Q_{\mathrm{u}}(\lambda)/Q_{\mathrm{u}}(\lambda = 0). \tag{25}
$$

In the spirit of this connection, we have

$$
Q_{\mathrm{u}}(\lambda) \triangleq \int\limits_{0}^{\infty} \prod_{\mu=1}^{N_{\mathrm{m}}''} \mathrm{d}z_\mu \left(\varepsilon + \sqrt{\varepsilon^2 + 2(N_{\mathrm{m}}' - 1)}\right)^{N_{\mathrm{m}}' - 1} \exp\left\{-\lambda \sum_{\mu=1}^{N_{\mathrm{m}}''} z_\mu - \tfrac{1}{2}\varepsilon^2 + \tfrac{1}{2}\varepsilon \sqrt{\varepsilon^2 + 2(N_{\mathrm{m}}' - 1)}\right\}. \tag{26}
$$

Setting $z_\mu = 2 z_\mu' + N_{\mathrm{m}} a/N_{\mathrm{m}}''$ gives $\varepsilon = -\Sigma z_\mu' \equiv -\mu$. If we employ exactly the same integration procedures as in part a) we find

$$
Q_{\mathrm{u}}(\lambda) \triangleq \exp\{-\lambda N_{\mathrm{m}} a\} \int\limits_{-N_{\mathrm{m}} a/2}^{\infty} \mathrm{d}\mu \exp\{-2\lambda \mu - \tfrac{1}{2}\mu^2 - \tfrac{1}{2}\mu \sqrt{\mu^2 + 2(N_{\mathrm{m}}' - 1)} + (N_{\mathrm{m}}'' - 1)
$$

$$
\cdot \ln(\mu + N_{\mathrm{m}} a/2) + (N_{\mathrm{m}}' - 1)\ln(\sqrt{\mu^2 + 2(N_{\mathrm{m}}' - 1)} - \mu)\} \equiv \exp\{-\lambda N_{\mathrm{m}} a\} \int\limits_{-N_{\mathrm{m}} a/2}^{\infty} \mathrm{d}\mu \exp\{f_{\mathrm{c}}(\mu)\}. \tag{27}
$$

One may try to employ again the method of steepest descent as in part a). Here, however, it will turn out to be more satisfying to use the mere knowledge as to when this method is successful, in order to obtain a closed expression for $Q(\lambda)$.

The reasoning proceeds as follows. The function $f_c(\mu)$ has for $N_m'' > 1$ one and only one relevant saddle point at $\mu_0 \in \{-N_m\,a/2; -N_m'\,a/2\}$ for all values of λ. For sufficiently large N_m' and N_m'' there exists an exceedingly sharp maximum on the real axis whose sharpness is measured by

$$f_c'''(\mu_0) = O\left((2/N_m'\,a)^2\right) + O\left((2/N_m''\,a)^2\right) \ll 1$$

which is the order of $|f_c''(\mu_0)|$. In this case all essential contributions to the integral come from the region around μ_0. For small N_m'' the method of steepest descent fails. But now the main contributions to the integral come from the region near $-N_m\,a/2$. Therefore, one can say that all points relevant in the integral (27) are contained in the interval $[-N_m\,a/2; -N_m'\,a/2 + \delta]$ with a reasonable choice of $\delta \lesssim N_m'\,a/2$. These μ therefore satisfy $\mu^2 \gtrsim (N_m'\,a/2)^2$. We now require $(N_m'\,a/2)^2 \gg 2\,N_m'$ or $N_m'\,a^2 \gg 1$ and then are allowed to substitute

$$f_c(\mu) \rightarrow \triangleq -2\,\lambda\,\mu + (N_m'-1)\,\ln(-\mu) + (N_m''-1)\,\ln(\mu + N_m\,a/2) \tag{28}$$

valid everywhere in the integration interval of relevance for the evaluation of Equation (27). We may even extend the upper limit of this interval from $-N_m'\,a/2 + \delta$ to zero and then obtain:

$$Q_u(\lambda) \triangleq \exp\{-\lambda\,N_m\,a\} \int_{-N_m\,a/2}^{0} d\mu\,(-\mu)^{N_m'-1}\,(\mu + N_m\,a/2)^{N_m''-1}\,\exp\{-2\,\lambda\,\mu\}\,. \tag{29}$$

This is a valid representation for $N_m'\,a^2 \gg 1$, $N_m' \gg 1$, and arbitrary values of N_m'' consistent with $N_m'' \ll N_m'^2$. For instance, in the case $a \approx 1$, $N_m = 10^6$, $[1; 9 \cdot 10^5]$ is an admissible range of N_m''-values. It will turn out that the final result is even more flexible than the above restrictions might indicate. Furthermore, N_m-values as low as 10 to 100 may be allowed to get reasonably correct results.

The further computational steps are straight forward. Substituting $\mu = (\mu' - N_m\,a)/2$ gives for Eq. (29)

$$Q_u(\lambda) \triangleq \int_0^{N_m\,a} d\mu'\,(N_m\,a - \mu')^{N_m'-1}\,\mu'^{N_m''-1}\,\exp\{-\lambda\,\mu'\}\,. \tag{30}$$

Comparing this equation with

$$Q(\lambda) = \int_0^\infty dI\,P(I)\,\exp\{-\lambda\,I\} \tag{31}$$

where $P(I)$ is the required distribution function for I, immediately gives

$$P(I) \triangleq (N_m\,a - I)^{N_m'-1}\,I^{N_m''-1}; I \leq N_m\,a; \triangleq 0, \ \text{otherwise.} \tag{32}$$

Specifying $Q_u(\lambda)$ by the expression (30) the correctly normalized $P(I)$ is found to be

$$P(I \leq N_m\,a) = (N_m\,a - I)^{N_m'-1}\,I^{N_m''-1}/Q_u(\lambda = 0)\,. \tag{33}$$

By an obvious substitution, $Q_u(\lambda = 0)$ may be expressed as

$$Q_u(\lambda = 0) = (N_m\,a)^{N_m-1} \int_0^1 dt\,t^{N_m''-1}(1-t)^{N_m'-1} = (N_m\,a)^{N_m-1}\,B(N_m'', N_m')\,. \tag{34}$$

B is the Beta-function which satisfies

$$1/B(N_m'', N_m') = N_m'\binom{N_m-1}{N_m''-1}\,. \tag{35}$$

Our final results therefore are

$$P(I \leq N_m\,a) = \frac{N_m'}{N_m\,a}\binom{N_m-1}{N_m''-1}\left(\frac{I}{N_m\,a}\right)^{N_m''-1}\left(1 - \frac{I}{N_m\,a}\right)^{N_m'-1} \tag{36}$$

$$Q(\lambda) = \left(1/B(N_m'', N_m')\right) \int_0^1 dt\,(1-t)^{N_m'-1}\,t^{N_m''-1}\,\exp\{-\lambda\,N_m\,a\,t\}\,. \tag{37}$$

This is the definition of the confluent hypergeometric function $_1F_1(N_m'', N_m, -\lambda\,N_m\,a)$ (cf. e. g. [20]):

$$Q(\lambda) = {}_1F_1(N_m'', N_m, -\lambda\,N_m\,a)\,. \tag{38}$$

4. Remarks on the Results

A closer look into a text-book on Mathematical Statistics (cf. e. g. [19]) reveals that our intensity distribution function $P(I)$ according to Eq. (36) is identical to the Beta-distribution in the interval $[0; N_m a]$ and that Eq. (38) is the well-known result for the moment generating function $Q(\lambda)$ associated with this distribution. We, therefore, can use some of the results known for the Beta-distribution.

Before we do so, we investigate Eq. (36) for $N_m'' = 1$. Because of $N_m' = N_m - 1 \gg 1$, $I = z$, and $(1 - z/(N_m a))^{N_m} \xrightarrow{z \lesssim a} \exp\{-z/a\}$ we get

$$P(z) = (1/a) \exp\{-z/a\}. \qquad (39)$$

This means, that any individual wave is thermally distributed (cf. [21]). This does not mean, however, that all waves are thermal. In fact, because of our starting Equation (13), only a very few waves can have intensity values $z \approx 0$ simultaneously. The high dimensional intensity distributions remain sharply peaked around $z_\mu = a$. We therefore expect, that $P(I)$ is similar to the distribution of N_m'' independent thermal waves for sufficiently small values of N_m'' while differences occur to this case for increasing N_m''. Another consequence of Eq. (39) is

$$\langle z \rangle = a. \qquad (40)$$

From our Eq. (10) this means that $(\tilde{u} \tilde{u}^*)_{op}$ is identical to the mean value $\langle \tilde{u}(\nu) \tilde{u}^*(\nu) \rangle$ and that a may be computed directly from the mean value. This is in contrast to the single wave case, where $\langle z \rangle \to a$ only for $a \gg 1$ [5].

Another point which requires discussion is the "cut off" of the distribution function $P(I)$ at $I = N_m a$. This is clearly a consequence of the approximations involved in the derivations and should be interpreted as follows: For $N_m \gg 1$ and $N_m'' < N_m$ the probabilities of intensity values $I \gtrless N_m a$ are exceedingly small and it makes no difference physically to neglect them at all. Note, that in our case the formal limit $N_m'' \to N_m$ which is not allowed because of the restrictions on N_m' and N_m'' given above leads to

$$Q(\lambda) \xrightarrow{N_m'' \to N_m} {}_1F_1(N_m, N_m, -\lambda N_m a)$$
$$= \exp\{-\lambda N_m a\}. \qquad (41)$$

This means, that the intensity distribution function $P(I)$ approaches $\delta(I - N_m a)$ which for $N_m \gg 1$

is quite an appropriate result. From $\langle z \rangle = a$ we get immediately

$$\langle I \rangle = N_m'' a \qquad (42)$$

using the definition Eq. (14) of I. This relation can also be derived from the moment generating function $Q(\lambda)$ using

$$\langle I^k \rangle = (-\partial/\partial\lambda)^k Q(\lambda)|_{\lambda=0} \qquad (43)$$

(cf. Appendix II). This constitutes a more basic justification for $\langle z \rangle = a$. For the mean square deviation one obtains

$$\langle (I - \langle I \rangle)^2 \rangle = [N_m'/(N_m + 1)] N_m'' a^2. \qquad (44)$$

In the case of N_m'' independent thermal waves which are described by a Gamma-distribution (cf. [19]) one gets

$$\langle (I - \langle I \rangle)^2 \rangle_{\text{thermal}} = N_m'' a^2. \qquad (45)$$

Evidently, the Beta-distribution is more sharply peaked around $(N_m'' a)$ than in the case of independent thermal waves. This makes the reasoning of Appendix I self consistent. For small $N_m'' = N_m - N_m' \ll N_m$ our earlier statement is substantiated that the waves behave like independent thermal waves.

5. Photon Counting Distribution

The most direct consequences of the statistical nature of waves can be revealed in measurements of photon counting distributions. If the waves themselves do not consist of photons, a light scattering experiment using a coherent light source (laser) can be performed in many cases. The statistical properties of the waves are then transformed to the scattered light beam. For our example of low frequency acoustic phonons, Brillouin scattering is the appropriate method for investigations of this kind [2].

The connection between the statistics of a classical wave field and the photon counting distribution $p(n)$ has been given by Mandel [22] and reads

$$p(n) = \int_0^\infty dW \, P(W) [(\nu_0 W)^n/n!] \exp\{-\nu_0 W\} \qquad (46)$$

where $W = \int_0^T dt \, I(t)$ is the integrated light intensity during the counting interval T. ν_0 describes the detector properties. In the case of a light scattering experiment when I is the wave intensity and not the scattered light intensity, ν_0 contains all the details

of the scattering process especially the scattering cross section. For the Brillouin scattering case this will be demonstrated in [18].

For arbitrary counting intervals T the distribution function $P(W)$ is not simply related to $P(I)$. If, however, T is small compared to the coherence times τ_c of the individual waves contributing to I, one has $W = T \cdot I$ and consequently:

$$p(n) = \int_0^\infty dI \, P(I) \left[(\nu_1 I)^n / n! \right] \exp\{ -\nu_1 I \} . \quad (47)$$

Due to the line narrowing of self sustained oscillations it is usually not difficult to achieve the case $T \ll \tau_c$. $\nu_1 \propto T$ need not be specified any further, since it disappears if the mean counting number $\langle n \rangle = \Sigma \, p(n) \, n$ is introduced. Note, that we take it as an a priori property of the detected signal, that the N_m'' individual waves contained in the signal contribute incoherently to the total intensity effective in the photo detection process.

Using the moment generating function $Q(\lambda)$, Equation (47) may be written:

$$p(n) = \frac{1}{n!} \left(-\nu_1 \frac{\partial}{\partial \lambda} \right)^n Q(\lambda) \Big|_{\lambda = \nu_1} . \quad (48)$$

Furthermore:

$$\langle n \rangle = -\nu_1 \, \partial Q(\lambda) / \partial \lambda \Big|_{\lambda = 0} \quad (49)$$

which in our case gives

$$\langle n \rangle = N_m'' \, a \, \nu_1 \quad (50)$$

as the desired relation between the measured quantity $\langle n \rangle$ and ν_1. Though the functional relations satisfied by the confluent hypergeometric function $_1F_1$ (see Appendix II) permit to evaluate Eq. (48) and express $p(n)$ in terms of $_1F_1$ again, this gives no insight into the structure of the counting distribution $p(n)$. It is more informative, to discuss the recurrence relation

$$p(n-1) = \frac{N_m'' (N_m + n - 1) + N_m \langle n \rangle}{N_m (N_m'' + n - 1)} \frac{n}{\langle n \rangle} p(n) - \frac{N_m'' \, n(n+1)}{N_m (N_m'' + n - 1) \langle n \rangle} p(n+1) \quad (51)$$

whose derivation is indicated in the Appendix II. For $N_m \gg (n, \langle n \rangle)$ Eq. (51) reduces to the recurrence relation

$$p(n-1) = \frac{N_m'' + \langle n \rangle}{N_m'' + n - 1} \frac{n}{\langle n \rangle} p(n) . \quad (52)$$

This is the recurrence relation for the photon counting distribution of N_m'' independent thermal waves. Its solution is a special case of Mandel's counting distribution [23]:

$$p(n) = \Gamma(n + N_m'') / \left(n! \, \Gamma(N_m'') \right) \frac{1}{(1 + \langle n \rangle / N_m'')^{N_m''}} \frac{1}{(1 + N_m'' / \langle n \rangle)^n} . \quad (53)$$

For $(n, \langle n \rangle, N_m'')$ comparable to N_m the second term in Eq. (51) cannot be neglected. To see this in more detail, we investigate the first cumulants of the counting distribution. They are derived in the usually way from the moment generating function $M_p(s)$ of the counting distribution

$$M_p(s) = \langle \exp\{s \, n\} \rangle = \sum_{n=0}^\infty p(n) \exp\{s \, n\} . \quad (54)$$

The relation of $M_p(s)$ to $M(s) \equiv Q(-s)$ immediately follows from Eq. (47) (cf. also [23, 24])

$$M_p(s) = M[\nu_1 (\exp\{s\} - 1)] \quad (55)$$

which allows to express the cumulants

$$k_m = (\partial / \partial s)^m \ln M_p(s) \big|_{s=0} \quad (56)$$

of the counting distribution $p(n)$ as a linear combination of the first m cumulants K_n of $P(I)$. In our case, one obtains for the first three cumulants:

$$k_1 \equiv \langle n \rangle ,$$

$$k_2 \equiv \langle (n - \langle n \rangle)^2 \rangle = \langle n \rangle + \left\{ \frac{N_m'}{N_m + 1} \right\} \frac{\langle n \rangle^2}{N_m''} ,$$

$$k_3 = \langle n \rangle + \left\{ \frac{N_m'}{N_m + 1} \right\} 3 \frac{\langle n \rangle^2}{N_m''}$$

$$+ \left\{ \frac{N_m' (N_m' - N_m'')}{(N_m + 1)(N_m + 2)} \right\} 2 \frac{\langle n \rangle^3}{N_m''^2} . \quad (57)$$

The expressions in curly brackets represent the differences of the cumulants of our counting distribution to that of N_m'' independent thermal waves, in

which case $\{ \} \equiv 1$. It is seen, that for N_m'' comparable to N_m marked differences appear provided the number $\langle n \rangle$ of mean counts is sufficiently large. The third term of k_3 even changes sign if N_m'' becomes greater than $N_m/2$ while k_2 shows that $p(n)$ is more sharply peaked around $\langle n \rangle$ than in the case of independent thermal waves.

With these remarks we conclude our discussion of the photon counting distribution. A complete insight into its structure apparently requires computer calculations e. g. based on the recurrence relation Equation (51).

6. Conclusions

We summarize the principle features of the interaction model we have investigated and the results we have obtained for it. Our starting point is a small frequency band of $N_m \gg 1$ equivalent waves which interact via their intensities. By means of a nonlinear amplification mechanism these waves are stabilized at a level far from thermal equilibrium. The stationary N_m-dimensional intensity distribution function, therefore, is sharply peaked around the corresponding operating point $\{a\}$ which differs from zero. We have determined analytically the distribution function for the incoherent superposition of a variable subset of $N_m'' < N_m$ waves. This was accomplished by an asymptotic expansion appropriate for $N_m \gg 1$. In praxi, N_m-values as low as 10 to 100 may be admissible. We found that the distribution function is given by the Beta-distribution associated with the two integers N_m'' and N_m, and defined in the interval $[0; N_m a]$. A discussion of this distribution revealed that the interactions thermalize each individual wave which without interactions would have exactly the same statistical properties as a single laser mode [5, 12]. The fact that in thermal equilibrium interactions thermalize waves is well-known from Statistical Mechanics and actually is almost an tautology. In our case, we see that this is true also for a state far from thermal equilibrium. There is, however, an important restriction to this statement. While in thermal equilibrium the thermalized waves tend to be statistically independent, strong correlations remain between the waves far from thermal equilibrium. In our case this becomes clear in the study of the photon counting distribution which for N_m'' comparable to N_m and sufficiently large mean counts $\langle n \rangle$ differs significantly from the counting distribution of N_m'' independent thermal waves.

Besides these more general aspects, one would like to utilize our results directly. A definite example known to the authors is the investigation of highly amplified vibrational noise in piezoelectric semiconductors by Brillouin scattering studies [2, 11]. In this case, all the presuppositions of our theory seem to be met.

Our results may provide a tool to investigate the strong interactions between the waves always appearing at a state far from thermal equilibrium. Especially it seems possible to determine experimentally the effective number N_m of interacting waves and the stabilization parameter a. Together with N_m'' which is the number of investigated waves these are the only parameters appearing in the theory.

Finally, we must concede that we have studied only the simplest problems of this kind namely those connected with stationary distribution functions. It seems, however, exceedingly complicated to investigate time correlations associated with transition probability functions in the case of many interacting waves. This is true even for the simplest interaction models such as the one we have discussed.

Appendix I

The roots of the equation $f'(\mu_1) = 0$ where $f(\mu_1)$ is given by Eq. (21) are

$$\mu_{10}^{(\pm)} = -(\lambda_1 + N_m' a/2) \pm \sqrt{(\lambda_1 - N_m' a/2)^2 + 2(N_m' - 1)}. \tag{I 1}$$

From its definition $\lambda_1 \in [-N_m'' a/2; \infty)$ and consequently

$$\mu_{10}^{(+)} \in (-N_m' a; \delta^{(+)} > N_m'' a] \qquad \text{and} \qquad \mu_{10}^{(-)} \in (-\infty; -N_m' a - \delta^{(-)})$$

where

$$\delta^{(-)} = 2(N_m' - 1)/(N_m a) - 4(N_m' - 1)^2/(N_m a)^3 > 0$$

because of $N_\mathrm{m}\, a \gg 1$. This means that the saddle point at $\mu_{10}^{(-)}$ does not contribute to the integral in Equation (20). A difficulty appears, however, if $\mu_{10}^{(+)}$ approaches $-N_\mathrm{m}'\, a$. The reliability of the method of steepest descent is estimated by

$$f'''(\mu_{10}) = 2\,(N_\mathrm{m}' - 1)/(\mu_{10} + N_\mathrm{m}'\, a)^3 \tag{I 2}$$

being much smaller than

$$-f''(\mu_{10}) = (N_\mathrm{m}' - 1)/(\mu_{10} + N_\mathrm{m}'\, a)^2 + 1/2\,. \tag{I 3}$$

Evidently for $\mu^{(+)} \to -N_\mathrm{m}'\, a$ this condition is violated. The problem is solved by noting, that this case is highly improbable. From Eq. (40) it follows, that $\langle \lambda_1 \rangle = 0$. In the most unfavourable case one expects the mean square deviation $\langle \lambda_1^2 \rangle$ to be of the order $1/4\, N_\mathrm{m}''\, a^2$. The region of λ_1 values which are of importance in the probability distribution P therefore is restricted to $\lambda_1 \lesssim \sqrt{N_\mathrm{m}''}\, n\, a/2$ where $n \gtrsim 1$. For $z_\mu < a$ no problem arises at all since $\mu_{10}^{(+)} > 0$. Difficulties of the type mentioned above appear, however, when $\lambda_1 > N_\mathrm{m}'\, a/2$. In this case $\mu_{10}^{(+)}$ may approach $-N_\mathrm{m}'\, a$. We therefore demand

$$2\,\lambda_1 \lesssim n\, a\, \sqrt{N_\mathrm{m}''} < N_\mathrm{m}'\, a \tag{I 4}$$

which essentially gives

$$N_\mathrm{m}'' \ll N_\mathrm{m}'^2\,. \tag{I 5}$$

This is the condition mentioned in Section 3. Dropping constant factors and unessential z_μ-dependencies (complying with the above restriction) the method of steepest descent yields Equation (23).

Appendix II

We summarize some of the consequences of Eq. (38) which follow from the properties of the confluent hypergeometric function (cf. e. g. [20]).

From a repeated application of

$$\frac{\partial}{\partial x}\, {}_1F_1(\alpha, \gamma, x) = \frac{\alpha}{\gamma}\, {}_1F_1(\alpha + 1, \gamma + 1, x)$$

and the property ${}_1F_1(x = 0) = 1$ one obtains for the k-th moment m_k

$$m_k = \langle I^k \rangle = \left(-\frac{\partial}{\partial \lambda}\right)^k Q(\lambda)\bigg|_{\lambda = 0} = (N_\mathrm{m}\, a)^k\, \frac{(N_\mathrm{m} - 1)!\, (N_\mathrm{m}'' + k - 1)!}{(N_\mathrm{m}'' - 1)!\, (N_\mathrm{m} + k - 1)!}\,. \tag{II 1}$$

Analogously, one has from Eq. (48)

$$p(n) = \left(\frac{N_\mathrm{m}}{N_\mathrm{m}''}\right)^n \frac{\langle n \rangle^n}{n!}\, \frac{(N_\mathrm{m} - 1)!\, (N_\mathrm{m}'' + n - 1)!}{(N_\mathrm{m}'' - 1)!\, (N_\mathrm{m} + n - 1)!}\, {}_1F_1(N_\mathrm{m}'' + n, N_\mathrm{m} + n, -N_\mathrm{m}\, a\, \nu_1)\,. \tag{II 2}$$

The normalized factorial moments

$$n^{(k)} \equiv \langle n(n-1)\ldots(n-k) \rangle / \langle n \rangle^k \tag{II 3}$$

are found to be given by

$$n^{(k)} = \left(\frac{N_\mathrm{m}}{N_\mathrm{m}''}\right)^k \frac{(N_\mathrm{m} - 1)!\, (N_\mathrm{m}'' + k - 1)!}{(N_\mathrm{m}'' - 1)!\, (N_\mathrm{m} + k - 1)!}\,. \tag{II 4}$$

They reflect solely the influence of N_m and N_m''. From the relation

$$\alpha\, {}_1F_1(\alpha + 1, \gamma, x) = (x + 2\,\alpha - \gamma)\, {}_1F_1(\alpha, \gamma, x) + (\gamma - \alpha)\, {}_1F_1(\alpha - 1, \gamma, x)$$

one derives

$${}_1F_1(N_\mathrm{m}'', N_\mathrm{m}, -N_\mathrm{m}\, a\, \lambda) = \frac{N_\mathrm{m}'}{N_\mathrm{m}}\, \frac{{}_1F_1(N_\mathrm{m}'' - 1, N_\mathrm{m}, -\lambda\, N_\mathrm{m}\, a)}{1 + a\, \lambda} - \frac{N_\mathrm{m}''}{N_\mathrm{m}}\, \frac{{}_1F_1(N_\mathrm{m}'' + 1, N_\mathrm{m}, -\lambda\, N_\mathrm{m}\, a)}{1 + a\, \lambda}\,. \tag{II 5}$$

For $N_\mathrm{m}'' \ll N_\mathrm{m}$ and consequently $N_\mathrm{m}' \to N_\mathrm{m}$ one thus obtains using ${}_1F_1(\alpha = 0) = 1$

$$Q(\lambda) \to [1/(1 + a\, \lambda)]^{N_\mathrm{m}''}\,. \tag{II 6}$$

This is the generating function of the incoherent superposition of N_m'' independent thermal waves [cf. Eq. (39)]. Since

$$(1/n!) \, (-\nu_1 \, \partial/\partial\lambda)^n \, (1 + a \, \lambda)^{-N_m''} \Big|_{\lambda=\nu_1} = (1/n!) \, (\nu_1 a)^n \, N_m'' \, (N_m''+1) \ldots (N_m''+n-1) \, (1 + a \, \nu_1)^{-(N_m+n)}$$

$$= \frac{\langle n \rangle}{n} \, \frac{N_m''+n-1}{N_m''+\langle n \rangle} \left[\frac{1}{(n-1)!} \left(-\nu_1 \, \frac{\partial}{\partial\lambda} \right)^{n-1} (1 + a \, \lambda)^{-N_m''} \Big|_{\lambda=\nu_1} \right]$$

the recurrence relation Eq. (52) is valid in this case. The corresponding recurrence relation for $p(n)$ derived from $Q(\lambda)$ according to Eq. (38) may be obtained from the differential equation satisfied by $_1F_1(\alpha, \gamma, x)$

$$x \, \frac{\partial^2}{\partial x^2} \, F + (\gamma - x) \, \frac{\partial}{\partial x} \, F - \alpha \, F = 0$$

which clearly permits to express the n-th derivative of $_1F_1(\alpha, \gamma, x)$ appearing in Eq. (48) for $p(n)$ through its $(n-1)$-th and $(n+1)$-th derivatives. The simple but lengthy calculation results in the recurrence relation Eq. (51) of Section 5.

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